Equivariantly uniformly rational varieties

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EQUIVARIANTLY UNIFORMLY RATIONAL VARIETIES

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ABSTRACT. We introduce equivariant versions of uniform rationality: given an algebraic group G, a G-variety is called G-uniformly rational (resp. G-linearly uniformly rational) if every point has a G-invariant open neighborhood equivariantly isomorphic to a G-invariant open subset of the affine space endowed with a G-action (resp. linear G-action). We establish a criterion for $\mathbb{G}_m$-uniform rationality of affine variety equipped with hyperbolic $\mathbb{G}_m$-action with a unique fixed point, formulated in term of their Altmann-Hausen presentation. We prove the $\mathbb{G}_m$-uniform rationality of Koras-Russel threefolds of the first kind and we also give example of non $\mathbb{G}_m$-uniformly rational but smooth rational $\mathbb{G}_m$-threefold associated to pairs of plane rational curves birationally non equivalent to a union of lines.

INTRODUCTION

A *uniformly rational* variety is a smooth variety for which every point has a Zariski open neighborhood isomorphic to an open subset of $\mathbb{A}^n$. A uniformly rational variety is in particular a smooth rational variety, however the converse is an open question [12, p.885]. Some partial results are known: in particular, the blowup of a uniformly rational variety along a smooth subvariety is still uniformly rational [5, 6]. In particular as every complete rational surface is obtained by sequences of point blowups in minimal rational surfaces which are themselves uniformly rational. This result implies that every smooth rational surface is uniformly rational.

The main goal of this article is to introduce two different equivariant versions of the definition of uniform rationality. These notions are stronger than the original definition, since we will also require open sets to be stable under certain group actions. We will use these notions in several ways. First of all, we show uniform rationality for some cases which were previously not known. Also, we will show that the corresponding conjecture on uniform rationality does not hold in the equivariant case. This leads to new examples for which uniform rationality without a group action is unknown. The main tool will be that, by considering the equivariant version, questions of rationality can be studied at the quotient level. The precise definitions of equivariant uniform rationality for an arbitrary algebraic group G, will be given in section 1.

These notions can be applied to actions of an algebraic torus $\mathbb{T}$. The complexity of the $\mathbb{T}$-action on a variety is given by the codimension of the general orbits. Thus, in the case of a faithful action, the complexity is $\dim(X) - \dim(\mathbb{T})$. In complexity zero, that is, for a toric variety we have a particular presentation of such varieties using cones in a lattice $N \simeq \mathbb{Z}^k$, and an equivalence between smooth rational toric varieties and uniformly rational toric varieties (see [11, p.29]). In fact, the open sets are $\mathbb{T}$-stable, and toric varieties satisfy the strongest condition of equivariant uniform rationality (*G-linearly uniformly rational*). In the case of complexity one a smooth rational $\mathbb{T}$-variety is still uniformly rational (see [19, Chapter 4]). A generalization of the presentation of toric varieties by cones in latices has been developed by Altmann and Hausen (see [1]). This presentation is realizable for a normal $\mathbb{T}$-variety of any complexity. The coordinate algebra of the variety endowed in addition with the grading induced by the $\mathbb{T}$-action can be re-obtained by a pair $(Y, \mathcal{D})$, where Y is a variety of dimension $\dim(X) - \dim(\mathbb{T})$ and $\mathcal{D}$ a so-called *polyhedral divisor* on Y , a generalization of cones. The $\mathbb{T}$-variety associated to a pair $(Y, \mathcal{D})$ is denoted by $\mathbb{S}(Y, \mathcal{D})$ and is its *A-H presentation*. Thanks to this presentation it has been proven in [3, Theorem 5] that any complete rational $\mathbb{T}$-variety of complexity one admits a covering by finitely many open charts isomorphic to the affine space $\mathbb{A}^n$. Again here, the proofs given by [3, Theorem 5] show that for this case, these varieties are $\mathbb{T}$-uniformly rational in the sense of section 1. This leads to consider the next natural step with $\mathbb{T}$-varieties of complexity two.

For the results of this article, we will apply these notions to *hyperbolic actions* of the multiplicative group on affine rational threefolds. Since the quotient is a rational surface, questions of equivariant uniform rationality can be treated by studying birational maps on rational surfaces (see section 3).

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The sketch of the article is as follows. The first section is devoted to the presentation of some equivariant definitions. The second section summarizes the A-H presentation in the particular case of hyperbolic $\mathbb{G}_m$-actions and explains how to use it in the problem of uniformly rational varieties. In section 3 we will apply these notions to a family of *$\mathbb{G}_m$-rational threefolds*. We show, for example, that all *Koras-Russell threefolds of the first kind*, and certain ones of the second kind are $\mathbb{G}_m$-linearly uniformly rational. These varieties has been studied by many authors [17, 15]. Although, these can be constructed by blowing-up $\mathbb{A}^3$, it is along singular subvariety, thus the results of [5, 6] do not allow to conclude (see subsection 3.1 for explicit equations). In a fourth section we find examples of smooth rational $\mathbb{G}_m$-threefolds which are not $\mathbb{G}_m$-linearly uniformly rational. Among these examples, we find some which are other Koras-Russell threefolds. We also describe a simpler example obtained by considering a cyclic cover of $\mathbb{A}^3$ along a divisor supported by an elliptic curve. It is not known if this variety is uniformly rational, without any group action.

Finally, in the last section, we show how the various notions of equivariant uniform rationality differ. We introduce a weaker equivariant version, and we give an example to show the difference with the previous versions.

One of the goals of this article is to show how these equivariant notions can lead to new insights on treating questions of uniform rationality.

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1. G-uniformly rational varieties: definitions and first properties

One begins this section by recalling the definition of *affine modification*. This construction requires to blowup a sub-scheme in a variety X and gives us a new variety X' and a birational morphism $X' \to X$. Thus using the result on the blowup of a uniformly rational variety along a smooth subvariety, it is then possible to construct many examples of uniformly rational variety in any dimension. In a second part the different equivariant definitions of uniformly rational variety are introduced.

Definition 1. [16, 9] Let (X, D, Z) be a triple consisting of a variety X, a principal divisor D on X and a closed sub-scheme Z. Then the affine modification of the variety X along D with center Z is the scheme $X' = \tilde{X}_Z \setminus D'$ where D' is the proper transform of D in the blow-up $\tilde{X}_Z \to X$ of X along Z.

A particular type of affine modification is the *hyperbolic modification* of a variety X with center at a closed sub-scheme $Z \subset X$ (see [25]): It is defined as the affine modification of $X \times \mathbb{A}^1$ with center $Z \times \{0\}$, and divisor $X \times \{0\}$. As an immediate corollary of [5, proposition 2.6], we obtain the following result:

Proposition 2. *Affine modifications and hyperbolic modifications of uniformly rational varieties along smooth centers are again uniformly rational.*

Example 3. Let $\mathbb{A}^n = \mathrm{Spec}(\mathbb{C}[x_1, \dots, x_n])$ and $I = (f, g)$ such that the sub-variety in $\mathbb{A}^n$ defined by I is smooth. Then the affine modification of $\mathbb{A}^n$ of center $I = (f, g)$ and divisor $D = \{f = 0\}$ is isomorphic to the subvariety $X' \subset \mathbb{A}^{n+1}$ defined by the equation:

$$\{g(x_1, \dots, x_n) - yf(x_1, \dots, x_n) = 0\} \subset \mathbb{A}^{n+1} = \mathrm{Spec}(\mathbb{C}[x_1, \dots, x_n, y]).$$

Is a uniformly rational variety.

We will now introduce analogous definitions, equivariant versions, adapted to G-varieties of the property to be uniformly rational.

Definition 4. Let X be a G-variety and $x \in X$.

i) We say that X is *G-linearly rational at the point x* if there exists a G-stable open neighborhood U_x of x, a linear representation of $G \to GL_n(V)$ and a G-stable open subset $U' \subset V \simeq \mathbb{A}^n$ such that U_x is equivariantly isomorphic to U'.

ii) We say that X is *G-rational at the point x* if there exists an open G-stable neighborhood U_x of x, an action of G on $\mathbb{A}^n$ and $U' \subset \mathbb{A}^n$ an open G-stable subvariety, such that U_x is equivariantly isomorphic to U'.

iii) A G-variety that is G-linearly rational (respectively G-rational) at each point is called *G-linearly uniformly rational (*respectively *G-uniformly rational).*

iv) A G-variety that admits a unique fixed point x_0 by the G-action is called *G-linearly rational* (respectively G-rational) if it is G-linearly rational (respectively G-rational) at x_0.

G-linearly uniformly rational or just G-uniformly rational varieties are certainly uniformly rational. The converse is trivially false: for instance the point $[1:0]$ in $\mathbb{P}^1$ does not admit any $\mathbb{G}_a$-invariant open neighborhood for the action defined by $t \cdot [u : v] \to [u + tv : v]$.

For an algebraic torus acting on a variety X by a theorem of Sumihiro [23] every point $x \in X$ admits a $\mathbb{T}$-invariant affine open neighborhood. Moreover, Gutwirth proved in [13] that the actions of $\mathbb{G}_m$ on $\mathbb{A}^2$ are linearisable. This result has been generalized by Bialynicki-Birula in [4], an effective algebriac torus action on $\mathbb{A}^n$ is linearisable for $\dim(\mathbb{T}) \geq n-1$. By another theorem (see [17]), a $\mathbb{G}_m$-action on $\mathbb{A}^3$ is linearisable. A consequence, for these varieties, the property of being $\mathbb{T}$-linearly uniformly rational is equivalent to the property to be $\mathbb{T}$-uniformly rational, namely:

Theorem 5. *For $\mathbb{T}$-varieties of complexity 0, 1 and for $\mathbb{G}_m$-threefolds the properties of being $\mathbb{T}$-linearly uniformly rational and $\mathbb{T}$-uniformly rational are equivalent.*

2. Hyperbolic $\mathbb{G}_m$-actions on smooth varieties

In this section we summarize the correspondence between smooth affine varieties X endowed with an effective hyperbolic $\mathbb{G}_m$-action and pairs $(Y, \mathcal{D})$ where Y is a variety, that we call *A-H quotient*, and $\mathcal{D}$ a so-called *segmental divisor* on Y. All the definitions and constructions will be adapted from that established in [1]. As an illustration we will explicit A-H presentation of certain affine threefolds, which will be used later on to study $\mathbb{G}_m$-uniform rationality.

2.1. **Segmental divisors.** Let $X = \mathrm{Spec}(A)$ be a smooth affine variety equipped with an effective $\mathbb{G}_m$-action. Its coordinate ring is $\mathbb{Z}$-graded in a natural way via $A = \bigoplus_{n\in\mathbb{Z}} A_n$ where $A_n := \{f \in A / f(\lambda \cdot x) = \lambda^n f(x)\}$. A $\mathbb{G}_m$-action said to be *hyperbolic* if there is at least one $n_1 < 0$ and one $n_2 > 0$ such that A_{n_1} and A_{n_2} are nonzero.

Definition 6. Given a smooth affine variety $X = \mathrm{Spec}(A)$ equipped with a hyperbolic $\mathbb{G}_m$-action, we denote by $Y_0(X)$ the categorical quotient $X//\mathbb{G}_m = \mathrm{Spec}(A_0)$ and by $\pi : Y(X) \to Y_0(X)$ the blow-up of $Y_0(X)$ with center at the closed subscheme defined by the ideal $\mathcal{I} = \langle A_d \cdot A_{-d}\rangle$, where $d > 0$ is chosen so that $\bigoplus_{n\in\mathbb{Z}} A_{dn}$ is generated by A_0 and $A_{\pm d}$. We call $Y(X)$ the *A-H quotient* of X. It is normal and projective over $Y_0(X) \simeq \mathrm{Spec}(\Gamma(Y(X), \mathcal{O}_{Y(X)}))$ (see [1]).

Remark. When X is smooth, by virtue of [24, Theorem 1.9, proposition 1.4], $Y(X)$ is isomorphic to the fiber product of the schemes $Y_{\pm}(X) = \mathrm{Proj}_{A_0}(\bigoplus_{n\in\mathbb{Z}_{\geq 0}} A_{\pm n})$ over $Y_0(X)$. In the case that X is a normal but singular variety, this fiber product may be reducible, and then $Y(X)$ coincides with one of its irreducible components (see [24]).

In the remainder of the article, we use the notation $\pi : \tilde{Y}_I \to Y$ to refer to the blow-up of an affine variety Y with center at the closed sub-scheme defined by the ideal $I \subset \Gamma(Y, \mathcal{O}_Y)$.

Definition 7. A *segmental divisor* $\mathcal{D}$ on an algebraic variety Y is a formal finite sum $\mathcal{D} = \sum [a_i, b_i] \otimes D_i$, where $[a_i, b_i]$ are closed intervals with rational bounds $a_i \leq b_i$ and D_i are prime Weil divisors on Y.

The set of all closed intervals with rational bounds admits a structure of abelian semigroup for the *Minkowski sum*, the Minkowski sum of two intervals $[a_i, b_i]$ and $[a_j, b_j]$ being the interval $[a_i + a_j, b_i + b_j]$.

For any $n \in \mathbb{Z}$, we have an evaluation from segmental divisors to the group of Weil $\mathbb{Q}$-divisors on Y defined by : $\mathcal{D}(n) = \sum q_i D_i$ where for all i, $q_i \in \mathbb{Q}$ is the lower bound of the interval $[na_i, nb_i]$.

Definition 8. A *proper-segmental divisor*, noted ps-divisor, $\mathcal{D}$ on Y is a segmental divisor where each D_i is an effective divisor and for every $n \in \mathbb{Z}$, $\mathcal{D}(n)$ satisfies the following properties:

1) $\mathcal{D}(n)$ is a $\mathbb{Q}$-Cartier divisor on Y.

2) $\mathcal{D}(n)$ is semi-ample, that is, for some $p \in \mathbb{Z}_{>0}$, Y is covered by complements of supports of effective divisors linearly equivalent to $\mathcal{D}(pn)$.

3) $\mathcal{D}(n)$ is big, that is, for some $p \in \mathbb{Z}_{>0}$, there exists an effective divisor D linearly equivalent to $\mathcal{D}(pn)$ such that $Y \setminus \mathrm{Supp}(D)$ is affine.

In the particular case of hyperbolic $\mathbb{G}_m$-action, the main Theorem of [1] can be reformulated as follows:

Theorem 9. *For any ps-divisor $\mathcal{D}$ on a normal semi-projective variety Y the scheme*

$$\mathbb{S}(Y, \mathcal{D}) = \mathrm{Spec}(\bigoplus_{n\in\mathbb{Z}} \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(n))))$$

is a normal affine variety of dimension $\dim(Y) + 1$ endowed with an effective hyperbolic $\mathbb{G}_m$-action, with A-H quotient isomorphic to Y. Conversely any normal affine variety X endowed with an effective hyperbolic $\mathbb{G}_m$-action is isomorphic to $\mathbb{S}(Y(X), \mathcal{D})$ for a suitable ps-divisor $\mathcal{D}$ on $Y(X)$.

Remark 10. Alternatively, see [8, 10], any finitely generated $\mathbb{Z}$-graded algebra A can be written in the from

$$A = \bigoplus_{n<0} \Gamma(Y, \mathcal{O}_Y(nD_-)) \oplus \Gamma(Y, \mathcal{O}_Y) \oplus \bigoplus_{n>0} \Gamma(Y, \mathcal{O}_Y(nD_+))$$

where (Y, D_+, D_-) is a triple consisting in a normal variety Y and suitable $\mathbb{Q}$-divisors D_+ and D_- on it. These two presentations are obtained from each other by setting $D_- = \mathcal{D}(-1)$,$D_+ = \mathcal{D}(1)$ and conversely $\mathcal{D} = \{1\}D_+ - [0,1](D_- + D_+)$.

Remark 11. A method to determine a ps-divisor $\mathcal{D}$ such that $X \simeq \mathbb{S}(Y, \mathcal{D})$ is to embed X as a $\mathbb{G}_m$-stable subvariety of a toric variety (see [1, section 11]). The calculation is then reduced to the toric case by considering an embedding in $\mathbb{A}^n$ with linear action of a torus $\mathbb{T}$ of dimension n for n sufficiently large. The inclusion of $\mathbb{G}_m \hookrightarrow \mathbb{T}$ corresponds to an inclusion of the lattice $\mathbb{Z}$ of one parameter subgroups of $\mathbb{G}_m$ in the lattice $\mathbb{Z}^n$ of one parameter subgroups of $\mathbb{T}$. We obtain the exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "F"'] & \mathbb{Z}^n \arrow[l, bend right, "s"'] \arrow[r, "P"'] & \mathbb{Z}^n/\mathbb{Z} \arrow[r] & 0 \end{tikzcd},$$

where F is given by the induced action of $\mathbb{G}_m$ on $\mathbb{A}^n$ and s is a section of F. The toric variety is determined by the first integral vectors v_i of the unidimensional cone generated by the i-th column vector of P considered as rays in the lattice $\mathbb{Z}^n$, and each v_i corresponds to a divisor. The support of D_i is the intersection between X and the divisor corresponding to v_i. In particular for each i, $\mathrm{Supp}(D_i)$ intersects the exceptional divisor of the blow-up $\pi : Y \to Y_0$. The segment associated to the divisor D_i is equal to $s(\mathbb{R}^n_{\geq 0} \cap P^{-1}(v_i))$. The section can be chosen such that the number of non zero coefficients is minimal.

Using this presentation and the fact that every $\mathbb{G}_m$-action on $\mathbb{A}^3$ is linearisable [17], we are able to characterize hyperbolic $\mathbb{G}_m$-actions on $\mathbb{A}^3$. Before stating the characterization, we can always assume that the linear hyperbolic action of $\mathbb{G}_m$ on $\mathbb{A}^3$ is given by two coordinates of positive weights and one of negative weight. Moreover, we remark that it can always be reduced to the case where $\mathbb{A}^3//\mathbb{G}_m \simeq \mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[u,v])$ using a cyclic cover of order equal to the negative weight along the zero locus of the coordinate of negative weight. In other words every model of $\mathbb{A}^3$ with hyperbolic action is the quotient by a cyclic finite group of a model whose $\mathbb{G}_m$-quotient is $\mathbb{A}^2$. After this, using [21] we obtain the exact A-H presentation of any $\mathbb{A}^3$ endowed with a hyperbolic $\mathbb{G}_m$-action.

Proposition 12. *Every $\mathbb{A}^3$ endowed with a hyperbolic $\mathbb{G}_m$-action is equivariantly isomorphic to a cyclic quotient of a $\mathbb{G}_m$-variety $\mathbb{S}(Y, \mathcal{D})$ with Y and $\mathcal{D}$ defined as follows:*
i) Y is isomorphic to the blow-up of $\mathbb{A}^2$ at the origin.
ii) $\mathcal{D}$ is of the form:

$$\mathcal{D} = \{p_1\} \otimes D_1 + \{p_2\} \otimes D_2 + [p_3, p_4] \otimes E,$$

with D_1, D_2 are strict transforms of the coordinate lines and E is the exceptional divisor of the blow-up.

Proof. Let $\mathbb{A}^3$ be endowed with a linear action of $\mathbb{G}_m$, applying the method describe previously we consider the exact sequence,

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "F"'] & \mathbb{Z}^3 \arrow[l, bend right, "s"'] \arrow[r, "P"'] & \mathbb{Z}^2 \arrow[r] & 0 \end{tikzcd}$$

where $F = {}^t(a, b, -c)$ and $P = \begin{pmatrix} u_{1,1} & 0 & u_{1,3} \\ 0 & u_{2,2} & u_{2,3} \end{pmatrix}$ with $u_{i,j} \geq 0$.

The toric variety generated by the vectors $\begin{pmatrix} u_{1,1} \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ u_{2,2} \end{pmatrix}$ and $\begin{pmatrix} u_{1,3} \\ u_{2,2} \end{pmatrix}$ corresponds to the blow-up of $\mathbb{A}^2$ at the origin. Two of this vectors correspond to the generators of $\mathbb{A}^2$ and thus the associated divisors are the strict transforms of the coordinate lines and the last corresponds to the exceptional divisor. To determine the coefficients, we used the formula $[a_i, b_i] = s(\mathbb{R}^n_{\geq 0} \cap P^{-1}(v_i))$.

$\square$

Example 13. [22, example 1.4.8] The presentation of $\mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[Y,Z,T])$ equipped with the hyperbolic $\mathbb{G}_m$-action $\lambda \cdot (Y,Z,T) = (\lambda^{-6}Y, \lambda^3 Z, \lambda^2 T)$ is $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)}, \mathcal{D})$ with $\pi : \tilde{\mathbb{A}}^2_{(u,v)} \to \mathbb{A}^2$ the blow-up of $\mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[u,v])$ at the origin and

$$\mathcal{D} = \left\{\frac{1}{2}\right\} D_1 + \left\{\frac{-1}{3}\right\} D_2 + \left[0, \frac{1}{6}\right] E,$$

where E is the exceptional divisor of the blow-up, D_1 and D_2 are the strict transforms of the lines $\{u=0\}$ and $\{u+v=0\}$ in $\mathbb{A}^2$ respectively. Indeed, $\mathbb{A}^3/\!/\mathbb{G}_m = \mathrm{Spec}(\mathbb{C}[YZ^2, YT^3]) \simeq \mathrm{Spec}(\mathbb{C}[u,v])$ and $d > 0$ in definition 6 has to be chosen so that $\bigoplus_{n\in\mathbb{Z}} A_{dn}$ is generated by A_0 and $A_{\pm d}$. This is the case if d is the least common multiple of the weights of the $\mathbb{G}_m$-action on $\mathbb{A}^3$. Thus $d = 6$ and $Y(X)$ is the the blow-up of $\mathbb{A}^2$ with center at the closed sub-scheme with ideal (YZ^2, YT^3), i.e. the origin with our choice of coordinates.

2.2. **Algebro-combinatorial criteria for $\mathbb{G}_m$-linear rationality.** Let X be a smooth rational variety of any dimension endowed with a hyperbolic $\mathbb{G}_m$-action which admits a fixed point x_0. We will develop in this subsection a strategy to decide whether the variety X is $\mathbb{G}_m$-rational. The goal is to find an appropriate $\mathbb{G}_m$-invariant open neighborhood X' of x_0, which will be $\mathbb{G}_m$-equivariantly isomorphic to an open $\mathbb{G}_m$-stable subvariety of $\mathbb{A}^n$. We will express this open subset as the complement of the zeros locus of a semi-invariant regular function $f \in \Gamma(X, \mathcal{O}_X)$.

Definition 14. [1, Definition 8.3] Let Y and Y' be normal semi-projective varieties and let $\mathcal{D}' = \sum [a'_i, b'_i] \otimes D'_i$ and $\mathcal{D} = \sum [a_i, b_i] \otimes D_i$ be ps-divisors on Y' and Y respectively.

i) Let $\varphi : Y \to Y'$ be a morphism such that $\varphi(Y)$ is not contained in $\mathrm{Supp}(D'_i)$ for any i. The *polyhedral pull-back* of $\mathcal{D}'$ is defined by $\varphi^*(\mathcal{D}') := \sum [a'_i, b'_i] \otimes \varphi^*(D'_i)$, where $\varphi^*(D'_i)$ is the usual pull-back of D'_i.

ii) Let $\varphi : Y \to Y'$ be a proper dominant map. The *polyhedral push-forward of* $\mathcal{D}$ is defined by $\varphi_*(\mathcal{D}) := \sum [a_i, b_i] \otimes \varphi_*(D_i)$, where $\varphi_*(D_i)$ is the usual push-forward of D_i.

Definition 15. Two pairs (Y_i, D_i) consisting of a variety Y_i and a Cartier divisor D_i on Y_i are called *birationally equivalent* if there exist a variety Z, and two proper birational morphisms $\varphi_i : Z \dashrightarrow Y_i$ such that the strict transforms $(\varphi_i^{-1})_*(D_i)$ coincide. For ps-divisors, we extend this notion in the natural way to pairs $(Y_i, \mathcal{D}_i)$ consisting of a semi-projective variety Y_i and a ps-divisor $\mathcal{D}_i$ on Y_i using the polyhedral push-forward defined above.

We consider hyperbolic $\mathbb{G}_m$-action with unique fixed point on smooth variety. By construction of ps-divisors $\mathcal{D}$, as in remark 11, we obtain that the divisor $\mathcal{D}$ in the A-H quotient (definition 6), has at most one exceptional divisor in the support. We denote by $\hat{\mathcal{D}}$ the divisor obtain by removing components whose support does not intersect the exceptional divisor.

Theorem 16. *Let X be a smooth rational variety endowed with a hyperbolic $\mathbb{G}_m$-action with a unique fixed point x_0. Then X is $\mathbb{G}_m$-rational if and only if there exists pairs $(Y, \mathcal{D})$ and $(Y', \mathcal{D}')$ such that X is equivariantly isomorphic to $\mathbb{S}(Y, \mathcal{D})$ and $\mathbb{S}(Y', \mathcal{D}')$ is equivariantly isomorphic to $\mathbb{A}^n$ endowed with a hyperbolic $\mathbb{G}_m$-action and such that $(Y, \hat{\mathcal{D}})$ and $(Y', \hat{\mathcal{D}}')$ are birationally equivalent.*

Proof. Suppose that X is $\mathbb{G}_m$-rational so that there exists an open $\mathbb{T}$-stable neighborhood U_{x_0} of x_0, an action of $\mathbb{T}$ on $\mathbb{A}^n$ and an open $\mathbb{T}$-stable subvariety $U' \subset \mathbb{A}^n$, and an equivariant isomorphism $\varphi : U_{x_0} \to U'$. We can always reduce to the case where U_{x_0} and U' are principal open sets. Indeed U_{x_0} is the complement of a closed stable subvariety of X determined by an ideal $\mathcal{I} = (f_0, \dots, f_k)$ where each $f_i \in \Gamma(X, \mathcal{O}_X)$ is semi-invariant. As U_{x_0} contains x_0, there exists at least one f_i which does not vanish at x_0. Denoting this function by f, U is contained in the principal open subset X_f. The restriction of φ to X_f induces an isomorphism between X_f and the principal open subset $\mathbb{A}^n_{f\circ\varphi^{-1}}$. Note that any non-constant semi-invariant function $f \in \Gamma(X, \mathcal{O}_X)$ such that $f(x_0) \neq 0$ is actually invariant. Indeed, letting w be the weight of f, we have $\lambda \cdot f(x_0) = \lambda^w f(x_0) = f(\lambda^{-1} \cdot x_0) = f(x_0)$ for all $\lambda \in \mathbb{G}_m$, and so $w = 0$.

Let $(Y, \mathcal{D})$ be the pair corresponding to X with $\mathcal{D}$ minimal in the sense defined in remark 11. We can identify every invariant function f on X non vanishing at x_0 with an element f of $\Gamma(Y, \mathcal{O}_Y)$ such that $V(f) \subset Y$ does not contain any irreducible components of $\mathrm{Supp}(\hat{\mathcal{D}})$. Indeed, by definition of the A-H quotient (definition 6), the center of the blow-up $\pi : Y \to Y_0$ corresponds to the image of the fixed point by the algebraic quotient morphism. By virtue of [2, proposition 3.3] the pair $(Y_f, \mathcal{D}_f = i^*(\mathcal{D}))$ where $i : Y_f \hookrightarrow Y$ is the canonical open embedding which describes the equivariant open embedding $j : X_f \simeq \mathbb{S}(Y_f, i^*(\mathcal{D})) \to X$. We have the following diagram:

$$\begin{tikzcd} X_f \arrow[r, hook, "j"] \arrow[d, "//\mathbb{G}_m"'] & X = \mathbb{S}(Y,\mathcal{D}) \arrow[d, "//\mathbb{G}_m"'] \\ Y_{0,f} \arrow[r, hook] & Y_0 \\ Y_f \arrow[u, "\pi_{|Y'}"] \arrow[r, hook, "i"] & Y = Bl_I(Y_0). \arrow[u, "\pi"'] \end{tikzcd}$$

A similar description holds for the invariant principal open subset $\mathbb{A}^n_{f\circ\varphi^{-1}}$ of $\mathbb{A}^n$ endowed with a hyperbolic $\mathbb{G}_m$-action. By [1, Corollary 8.12.] X_f and $\mathbb{A}^n_{f\circ\varphi^{-1}}$ are equivariantly isomorphic if and only if there are projective birational morphisms $\sigma_1 : Y_f \to Y''$ and $\sigma_2 : Y'_{f\circ\varphi^{-1}} \to Y''$ and a ps-divisor $\mathcal{D}''$ with $\mathcal{D} \cong \psi_1^*(\mathcal{D}'')$ and $\mathcal{D}'_{f\circ\varphi^{-1}} \cong \psi_2^*(\mathcal{D}'')$. The construction of A-H quotients using definition 6 ensures that there exists a unique exceptional divisor. Since σ_1 is projective and birational it either contracts the unique exceptional divisor in Y, or it is an isomorphism. However if σ_1 is a contraction, $\mathbb{S}(Y'',\mathcal{D}'')$ is not equivariantly isomorphic to X_f. Therefore σ_1 is an isomrphism. The same holds for σ_2. Since $\mathcal{D}_f$ and $\mathcal{D}'_{f\circ\varphi^{-1}}$ are minimal the pairs $(Y_f,\mathcal{D}_f)$ and $(Y'_{f\circ\varphi^{-1}},\mathcal{D}'_{f\circ\varphi^{-1}})$ are equal up to isomorphism and so the pairs $(Y,\hat{\mathcal{D}})$ and $(Y',\hat{\mathcal{D}}')$ are birationally equivalent. This yields:

$$\begin{tikzcd} \mathbb{S}(Y',\mathcal{D}') = \mathbb{A}^n \arrow[d, "//\mathbb{G}_m"'] & \mathbb{A}^n_{f\circ\varphi^{-1}} \simeq X_f \arrow[l, hook', "j'"'] \arrow[r, hook, "j"] \arrow[d, "//\mathbb{G}_m"'] & X = \mathbb{S}(Y,\mathcal{D}) \arrow[d, "//\mathbb{G}_m"'] \\ Y'_0 & Y'_{0,f\circ\varphi^{-1}} \simeq Y_{0,f} \arrow[l, hook'] \arrow[r, hook] & Y_0 \\ Y' \arrow[u] & Y'_{f\circ\varphi^{-1}} \simeq Y_f \arrow[l, hook', "i'"'] \arrow[r, hook, "i"] \arrow[u] & Y. \arrow[u] \end{tikzcd}$$

In the opposite direction assume that X is equivariantly isomorphic to $\mathbb{S}(Y,\mathcal{D})$ and $\mathbb{S}(Y',\mathcal{D}')$ is equivariantly isomorphic to $\mathbb{A}^n$ endowed of an hyperbolic $\mathbb{G}_m$-action and such that $(Y,\hat{\mathcal{D}})$ and $(Y',\hat{\mathcal{D}}')$ are birationally equivalent. We can assume that we have $g \in \Gamma(Y,\mathcal{O}_Y)$ an isomorphism $\phi : V_g \to V_{g\circ\phi^{-1}}$ defined on a principal open set V_g of Y containing the exceptional divisor. The function $g \in \Gamma(Y,\mathcal{O}_Y)$ can be identified with an invariant function on X and by virtue of [2, proposition 3.3] again the pair $(Y_g,\mathcal{D}_g)$ describes the equivariant open embedding $X_g \simeq \mathbb{S}(Y_g,\mathcal{D}_g) \to X$. In the same way the pair $(Y_{g\circ\phi^{-1}},\mathcal{D}_{g\circ\phi^{-1}})$ describes the equivariant open embedding $\mathbb{A}^n_{g\circ\phi^{-1}} \simeq \mathbb{S}(Y'_{g\circ\phi^{-1}},\mathcal{D}'_{g\circ\phi^{-1}}) \to \mathbb{A}^n$, which gives the result. ☐

3. Applications to $\mathbb{G}_m$-threefolds

In the particular case of affine threefolds, $\mathbb{G}_m$-linear uniform rationality is reduced (by the previous section) to a problem of birational geometry in dimension 2. Indeed, using Theorem 16, the question may then be considered at the level of the quotients which are rational semi-projective surfaces. Furthermore the hyperbolic $\mathbb{G}_m$-actions on $\mathbb{A}^3$ are classified (proposition 12) in terms of ps-divisors.

3.1. **$\mathbb{G}_m$-linear uniform rationality.** In this subsection we will prove that some hypersurfaces of $\mathbb{A}^4$ are $\mathbb{G}_m$-linearly uniformly rational. In particular every *Koras-Russell threefold of the first kind* X is $\mathbb{G}_m$-linearly uniformly rational. These varieties are defined by equations of the form:

$$\{x + x^d y + z^{\alpha_2} + t^{\alpha_3} = 0\} \subset \mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x,y,z,t]),$$

where $d \geq 2$, and α_2 and α_3 are coprime. These smooth rational varieties are endowed with hyperbolic $\mathbb{G}_m$-actions with algebraic quotients isomorphic to $\mathbb{A}^2/\mu$ where μ is a finite cyclic group. They have been classified by Koras and Russell, in the context of the linearization problem for $\mathbb{G}_m$-actions on $\mathbb{A}^3$ [17].

These threefolds can be viewed as affine modifications of $\mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x,z,t])$ along the principal divisor D_f with center $I = (f,g)$ where $f = -x^d$ and $g = x + z^{\alpha_2} + t^{\alpha_3}$. But since I is supported on the cuspidal curve included in the plane $\{x = 0\}$ and given by the equation : $C = \{x^d = z^{\alpha_2} + t^{\alpha_3} = 0\}$ (see [25]) their uniformly rationality do not follow straight from corollary 2.

3.1.1. *General construction of $\mathbb{G}_m$-uniformly rational threefolds.* Here we give a general criterion to decide the $\mathbb{G}_m$-uniform rationality of certain threefolds, arising as stable hypersurfaces of $\mathbb{A}^4$ endowed with a linear $\mathbb{G}_m$-action. Since X is rational, its A-H quotient $Y(X)$ is also rational. The aim is to use the notion of birational equivalence of ps-divisor to construct an isomorphism of $\mathbb{G}_m$-stable open set of the variety X with an corresponding stable open subset f $\mathbb{A}^3$. By Theorem 16, the technique is to consider a well chosen sequence birational transformations to obtain a birational map in $\mathbb{A}^2$ that will send the ps-divisor corresponding to the threefolds on the coordinates axes of $\mathbb{A}^2$.

Let $p \in \mathbb{C}[v]$ be a polynomial in one variable of degree k, let α_2, α_3 and d be integers such that d and α_3 are coprime, likewise α_2 and α_3 are coprime. Let X be a hypersurface in $\mathbb{A}^4 = \operatorname{Spec}(\mathbb{C}[x, y, z, t])$ defined by one of the following equations:

$$\begin{cases} y^d z^{\alpha_2} + y t^{\alpha_3} + p(xy) = 0 & \text{if } p(0) \neq 0 \\ y^{d-1} z^{\alpha_2} + t^{\alpha_3} + p(xy)/y = 0 & \text{if } p(0) = 0. \end{cases}$$

Every such X is endowed with a hyperbolic $\mathbb{G}_m$-action induced by the linear action on $\mathbb{A}^4$ defined by $\lambda \cdot (x, y, z, t) = (\lambda^{\alpha_2\alpha_3} x, \lambda^{-\alpha_2\alpha_3} y, \lambda^{d\alpha_3} z, \lambda^{\alpha_2} t)$.

Theorem 17. *With the notation above we have:*

1) X is equivariantly isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v^d)}, \mathcal{D})$ with

$$\mathcal{D} = \left\{ \frac{a}{\alpha_2} \right\} D_1 + \left\{ \frac{b}{\alpha_3} \right\} D_2 + \left[0, \frac{1}{\alpha_2\alpha_3} \right] E,$$

where E is the exceptional divisor of the blow-up $\pi : \tilde{\mathbb{A}}^2_{(u,v^d)} \to \mathbb{A}^2 = \operatorname{Spec}(\mathbb{C}[u, v])$, D_1 and D_2 are the strict transforms of the curves $L_1 = \{u = 0\}$ and $L_2 = \{u + p(v) = 0\}$ in $\mathbb{A}^2$ respectively, and $(a, b) \in \mathbb{Z}^2$ are chosen such that $ad\alpha_3 + b\alpha_2 = 1$.

2) X is smooth and has a fixed point if and only if $D = L_1 + L_2$ in $\mathbb{A}^2$ is a SNC divisor and L_2 contains the origin. This is equivalent to the condition that $p(0) = 0$ and p has simple roots.

3) Under these conditions, X is $\mathbb{G}_m$-linearly rational at $(0, 0, 0, 0)$.

Proof. 1) To determine the A-H presentation of the variety X we consider the following exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "F"'] & \mathbb{Z}^4 \arrow[l, bend right, "s"'] \arrow[r, "P"'] & \mathbb{Z}^3 \arrow[r] & 0 \end{tikzcd},$$

where $F = {}^t(\alpha_2\alpha_3, -\alpha_2\alpha_3, d\alpha_3, \alpha_2)$, $P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & d & \alpha_2 & 0 \\ 0 & 1 & 0 & \alpha_3 \end{pmatrix}$ and $s = (0, 0, a, b)$ chosen such that $ad\alpha_3 + b\alpha_2 = 1$.

We consider the fan generated by $\{v_i\}_{i=1,..,4}$ where v_i is the first integral vector of the unidimensinal cone generated by the i-th colum vector of P. This fan corresponds to the blow-up of $\mathbb{A}^3 = \operatorname{Spec}(\mathbb{C}[u, v, w])$ along the sub-scheme with defining by the ideal $I = (u, v^d, v^{d-1}w, \ldots, vw^{d-1}, w^d)$, as a toric variety.

Then the variety Y corresponds to the strict transform by $\pi : \tilde{\mathbb{A}}^3_I \to \mathbb{A}^3 \simeq \mathbb{A}^4 /\!/ \mathbb{G}_m$ of the surface $\{u + w + p(v) = 0\} \simeq \operatorname{Spec}(\mathbb{C}[u, v])$, that is, $Y \simeq \tilde{\mathbb{A}}^2_{(u,v^d)}$ (see [21, section 3.1]).

The ps-divisor $\mathcal{D}$ is of the form $\left\{ \frac{a}{\alpha_2} \right\} D_1 + \left\{ \frac{b}{\alpha_3} \right\} D_2 + \left[0, \frac{1}{\alpha_2\alpha_3} \right] E$, where D_1 corresponds to the restriction to Y of the toric divisor given by the ray v_3 and D_2 corresponds to the restriction to Y of the toric divisor given by the ray v_4, that is, the strict transforms of the curves $\{u = y^d z^{\alpha_2} = 0\}$ and$\{w = y t^{\alpha_3} = -u - p(v) = 0\}$ in $\mathbb{A}^2$ respectively. The divisor E corresponds to the divisor given by v_2, that is, the exceptional divisor.

2) The only possible fixed point for X is the origin of $\mathbb{A}^4$ which is possible if and only if $p(0) = 0$. In this case the equation of X takes the form :

$$y^{d-1} z^{\alpha_2} + t^{\alpha_3} + x \prod_{i=1}^{k} (xy + \alpha_i) = 0,$$

and using the Jacobian criterion, we conclude that X is smooth if and only if $\alpha_i \neq \alpha_j$ for $i \neq j$.

3) Let $D = L_1 + L_2 \subset \mathbb{A}^2_{(u,v)}$ and consider the embedding in $\mathbb{P}^2_{(u:v:w)}$ (see Figure 3.1). After a sequence of elementary birational transformations we have a divisor in the k-th Hirzebruch surface $\mathbb{F}_k = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k))$ in which the proper transform of L_2 is a smooth curve intersecting the section of negative self intersection transversaly (see Figure 3.2). The second step is the blow-up of all the intersection points between $\bar{L_1}$ and $\bar{L_2}$

except the point corresponding to the origin in $\mathbb{A}^2$, followed by the contraction of the proper transform of the fiber passing throught each points of the blowup (see Figure 3.3). The final configuration is then the Hirzebruch surface $\mathbb{F}_1$ in which the proper transforms of $\bar{L}_1$ and $\bar{L}_2$ have self intersection 1 and intersect each other in a single point. Then $\mathbb{P}^2$ and the desired divisor are obtained from $\mathbb{F}_1$ by contraction of the negative section (see Figure 3.4).

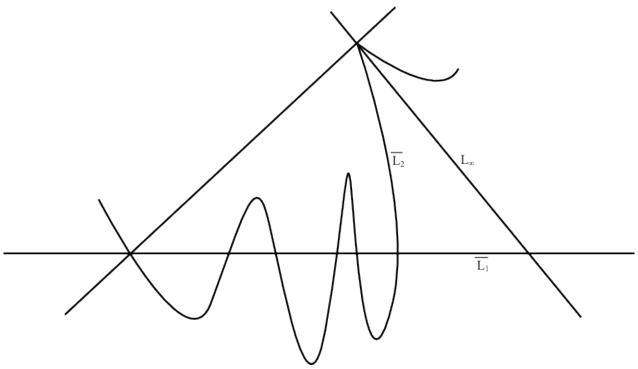

FIGURE 3.1. Embedding in $\mathbb{P}^2$ of the divisor in $\mathbb{P}^2$.

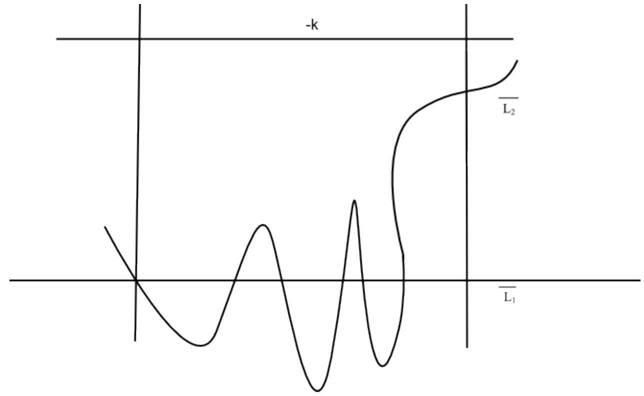

FIGURE 3.2. First sequence of blow-ups and contractions, to obtain a smooth normal crossing divisor divisor in $\mathbb{F}_k$.

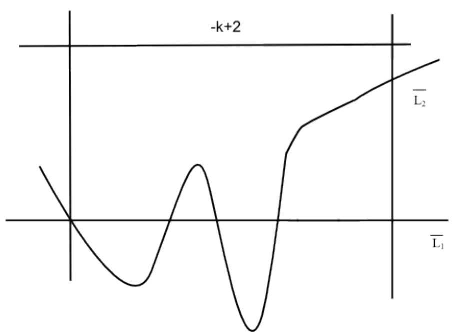

FIGURE 3.3. Intermediate step, resolution of the crossings, to obtain a divisor in $\mathbb{F}_{k-2}$.

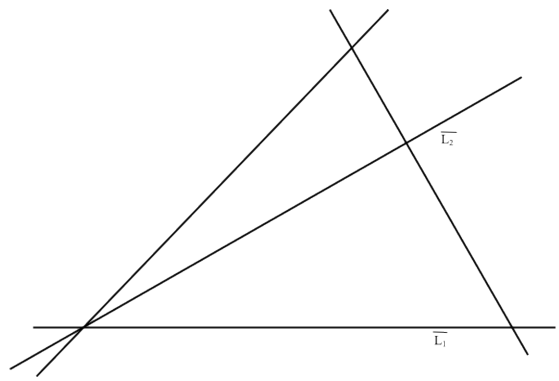

FIGURE 3.4. Final resolution, to obtain a divisor in $\mathbb{P}^2$

By Theorem 16 this resolution gives a $\mathbb{G}_m$-equivariant isomorphism between an open neighborhood of the origin in X and on open neighborhood of the origin in $\mathbb{A}^3$.

Let $D = L_1 + L_2$ in $\mathbb{A}^2$ where $L_1 = \{u = 0\}$ and $L_2 = \{u + v\prod_{i=1}^{k}(v + \alpha_i) = 0\}$, and let $\pi : \tilde{\mathbb{A}}^2_{(u,v+\alpha_k)} \to \mathbb{A}^2$ the blow-up of $\mathbb{A}^2$ along the sub-scheme given by the ideal $(u, v + \alpha_k)$. The equation of the strict transform of D in one of the charts is given by $D' = L_1' + L_2'$, where $L_1' = \{u' = 0\}$ and $L_2' = \{u' + v'\prod_{i=1}^{k-1}(v' + \alpha_i) = 0\}$. By induction, D is birationally equivalent to $D'' = L_1'' + L_2''$, where $L_1'' = \{u'' = 0\}$ and $L_2'' = \{u'' + v'' = 0\}$. Then by Theorem 12, and [1, Corollary 8.12.] X is $\mathbb{G}_m$-linearly rational. $\square$

More precisely, we will explicit the birational map on the A-H quotients of X and $\mathbb{A}^3$.

Let $p(v) = v(1 + g(v))$ be the polynomial which appears in the previous theorem, and let ϕ be the birational map defined by:

$$\phi : (u, v) \to (-u'(g(v' + u') + 1), v' + u').$$

Its inverse is defined by

$$\phi^{-1} : (u', v') \to (-\frac{u}{1 + g(v)}, v + \frac{u}{1 + g(v)}).$$

Then $\phi(u + p(v)) = v'(g(v' + u') + 1)$ thus we have :

$$Y(\mathbb{A}^n) \xleftarrow{\quad i \quad} Y' = \tilde{\mathbb{A}}^2_{(u,v^d)} \setminus V(1+g(v)) \simeq \tilde{\mathbb{A}}^2_{(u',v'^d)} \setminus V(g(v'+u')+1) \xhookrightarrow{\quad i' \quad} Y(X)$$

and $i : Y' \hookrightarrow \tilde{\mathbb{A}}^2_{(u,v^d)}$, then $\mathbb{S}(Y', i^*(\mathcal{D})) = U$ is an open neighborhood of the fixed point, which is moreover equivariantly isomorphic to an open of $\mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[Y,Z,T])$ endowed of the hyperbolic $\mathbb{G}_m$-action defined by $\lambda \cdot (Y,Z,T) = (\lambda^{-\alpha_2\alpha_3}Y, \lambda^{d\alpha_3}Z, \lambda^{\alpha_2}T)$ using Theorem 12.

Remark 18. The same process can be applied even if $L_1 + L_2$ is not a smooth normal crossing divisor under the condition that the crossing at the origin is transversal. This corresponds to the case where X is not smooth, but the fixed point is smooth and admits an open $\mathbb{G}_m$-stable neighborhood isomorphic to a $\mathbb{G}_m$-stable neighborhood of the fixed point of $\mathbb{A}^3$ with hyperbolic action. In other words X is $\mathbb{G}_m$-linearly rational, but of course not uniformly rational, since X has singularities.

3.1.2. *Explicit construction of $\mathbb{G}_m$-uniformly rational threefolds.* Considering the previous sub-section, it is now possible to explicit particular hypersurfaces of $\mathbb{A}^4$ which are $\mathbb{G}_m$-uniformly rational.

Proposition 19. *The following hypersurfaces in $\mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x,y,z,t])$ are $\mathbb{G}_m$-linearly rational:*

$$X_1 = \{x + x^k y^{k-1} + z^{\alpha_2} + t^{\alpha_3} = 0\}$$

$$X_2 = \{x + y^{d-1}(x^d + z^{\alpha_2}) + t^{\alpha_3} = 0\}.$$

Proof. Applying the Theorem 17, then X_1 corresponds to the choice $d = 1$ and $p(v) = v + v^k$, and X_2 corresponds to the choice $d \geq 2$ and $p(v) = v + v^d$.

1) Let $X_1 \setminus V(1 + (xy)^{d-1})$ and $\mathbb{A}^3 \setminus V(1 + (YZ^{\alpha_2} + YT^{\alpha_3})^{d-1})$, then the application ψ is given by:

$$\psi : \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} Y \\ Z \\ T \end{pmatrix} = \begin{pmatrix} \frac{-y}{1+(xy)^{d-1}} \\ z \\ t \end{pmatrix}$$

and its inverse ψ^{-1} is given by:

$$\psi^{-1} : \begin{pmatrix} Y \\ Z \\ T \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -\frac{Z^{\alpha_2}+T^{\alpha_3}}{1+(YZ^{\alpha_2}+YT^{\alpha_3})^{d-1}} \\ -Y(1+(YZ^{\alpha_2}+YT^{\alpha_3})^{d-1}) \\ Z \\ T \end{pmatrix}$$

2) Let $X_2 \setminus V(1 + (xy)^{d-1})$ and $\mathbb{A}^3 \setminus V(1 + (Y^d Z^{\alpha_2} + YT^{\alpha_3})^{d-1})$, then the application ψ is given by:

$$\psi : \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} Y \\ Z \\ T \end{pmatrix} = \begin{pmatrix} \frac{-y}{1+(xy)^{d-1}} \\ z \\ t \end{pmatrix}$$

and its inverse ψ^{-1} is given by:

$$\psi^{-1} : \begin{pmatrix} Y \\ Z \\ T \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -Y^{d-1}Z^{\alpha_2} - \frac{T^{\alpha_3}}{1+(Y^dZ^{\alpha_2}+YT^{\alpha_3})^{d-1}} \\ -Y(1+(YZ^{d\alpha_2}+YT^{\alpha_3})^{d-1}) \\ Z \\ T \end{pmatrix}$$

□

Theorem 20. *All the Koras-Russell of the first kind $\{x + x^k y + z^{\alpha_2} + t^{\alpha_3} = 0\}$ in $\mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x,y,z,t])$ are $\mathbb{G}_m$-linearly uniformly rational.*

Proof. Let $X = \{x + x^k y + z^{\alpha_2} + t^{\alpha_3} = 0\}$ be a Koras-Russell threefold of the first kind and let $\mathcal{U} \subset X$ be the open given by x is not zero. Then X is $\mathbb{G}_m$-linearly rational at every point in $\mathcal{U}$: if $x \neq 0$ then

$$\mathbb{C}[x, x^{-1}, y, z, t]/(x + x^k y + z^{\alpha_2} + t^{\alpha_3}) \simeq \mathbb{C}[x, x^{-1}, z, t].$$

By the corollary 19 we have an explicit $\mathbb{G}_m$-equivariant isomorphism between an open neighborhood of the fixed point included in $X_1 = \{x + x^k y^{k-1} + z^{\alpha_2} + t^{\alpha_3} = 0\}$ and en open subset of $\mathbb{A}^3$. In addition X_1 admits an action of the cyclic group μ_{k-1} given by $\epsilon \cdot (x, y, z, t) \to (x, \epsilon y, z, t)$ and the quotient for this action and the isomorphism commute indeed the orbits of the cyclic group are included in the orbits of the $\mathbb{G}_m$-action. In this case the quotient of $\mathbb{A}^3$ for the action of μ_{k-1} is still isomorphic to $\mathbb{A}^3$. Since $X_1//\mu_{k-1} \simeq X$, the map ψ which is $\mathbb{G}_m$-equivariant in the corollary 19 gave the appropriate map ψ_{k-1} that is $\mathbb{G}_m$-equivariant substituting y to y^{k-1}:

$$\begin{tikzcd}
X_1 \setminus V(1 + (xy)^{k-1}) \arrow[r, "\psi"] \arrow[d, "//\mu_{d-1}"] & \mathbb{A}^3 \setminus V(1 + (YZ^{\alpha_2} + YT^{\alpha_3})^{k-1}) . \arrow[d, "//\mu_{d-1}"] \\
X \setminus V(1 + yx^{k-1}) \arrow[r, "\psi_{d-1}"] & \mathbb{A}^3 \setminus V(1 + Y(Z^{\alpha_2} + T^{\alpha_3})).
\end{tikzcd}$$

Thus the principal $\mathbb{G}_m$-stable open subsets $\mathcal{U} = X_x$ and $\mathcal{V} = X_{1+yx^{d-1}}$ form a covering of X by $\mathbb{G}_m$-uniformly rational varieties.

□

Proposition 21. *Koras-Russell threefolds of the second kind given by the equations*

$$X = \{x + y(x^d + z^{\alpha_2})^l + t^{\alpha_3} = 0\},$$

in $\mathbb{A}^4 = \operatorname{Spec}(\mathbb{C}[x, y, z, t])$ *with* $l = 1$ *or* $l = 2$ *or* $d = 2$ *are* $\mathbb{G}_m$*-linearly uniformly rational.*

Proof. In the case $l = 1$ we consider the variety:

$$X_2 = \{x + y^{d-1}(x^d + z^{\alpha_2}) + t^{\alpha_3} = 0\},$$

given in the corollary 19. The conclusion follows by applying exactly the same method as in Theorem 20 using the action of the cyclic group μ_{d-1} on X_2 via $\epsilon \cdot (x, y, z, t) \to (x, \epsilon y, z, t)$.

In the case where $l = 2$ or $d = 2$, another argument is used. Let $X_{d-1} = \{x + y^{dl-1}(x^d + z^{\alpha_2})^l + t^{\alpha_3} = 0\} \to X$ be the cyclic cover of order $dl - 1$ of X branched along the divisor $\{y = 0\}$.

By showing that X_{d-1} is $\mathbb{G}_m$-linearly rational then one can explicit a birational map between X and $\mathbb{A}^3$. This map will be an equivariant isomorphism between an open subset of X containing the fixed point and an open subset of $\mathbb{A}^3$. The A-H presentation of X_{d-1}(see [21]) is $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v^d)}, \mathcal{D})$ with:

$$\mathcal{D} = \left\{\frac{a}{\alpha_2}\right\} D_{\alpha_3} + \left\{\frac{b}{\alpha_3}\right\} D_{\alpha_2} + \left[0, \frac{1}{\alpha_2\alpha_3}\right] E,$$

where E is the exceptional divisor of the blow-up $\pi : \tilde{\mathbb{A}}^2_{(u,v^d)} \to \mathbb{A}^2 \simeq \operatorname{Spec}(\mathbb{C}[u, v]) \simeq \operatorname{Spec}(\mathbb{C}[y^d z^{\alpha_2}, yx])$, and where D_{α_2} and D_{α_3} are the strict transforms of the curves $L_1 = \{v + (u + v^d)^l) = 0\}$ and $L_2 = \{u = 0\}$ in $\mathbb{A}^2 = \operatorname{Spec}(\mathbb{C}[u, v])$ respectively, $(a, b) \in \mathbb{Z}^2$, being chosen so that $ad\alpha_3 + b\alpha_2 = 1$.

First of all, variables l and d can be exchanged, just considering the automorphism of $\mathbb{A}^2 = \operatorname{Spec}(\mathbb{C}[u, v])$ which send u on $u - (v - u^l)^d$ and v on $v - u^l$. Then $v + (u + v^d)^l)$ is sent on v. From now it will be assumed that $l = 2$.

Let φ be the birational endomorphism of $\mathbb{A}^2 = \operatorname{Spec}(\mathbb{C}[u, v])$ defined by sending u on $\frac{u(1+(v-u^2)^{2d-1})}{1-u(v-u^2)^{d-1}}$ and v on $v - u^2$, this application allows us to show that $D = L_1 + L_2$ is birationally equivalent to $D' = \{uv = 0\}$. Thus X_{d-1} is $\mathbb{G}_m$-linearly rational. Moreover the application φ is μ_{2d-1}-equivariant, considering the action of μ_{2d-1} given by $\epsilon \cdot (u, v) \to (\epsilon^d u, \epsilon v)$. The desired result is now obtained by the same technique as in Theorem 20. □

We will give an example of $\mathbb{G}_m$-variety X with a unique fixed point, and equivariantly isomorphic to $\mathbb{S}(Y, \mathcal{D})$ which is $\mathbb{G}_m$-linearly rational and such that irreducible components do not all intersect the exceptional divisor, that is, $\hat{\mathcal{D}} \neq \mathcal{D}$.

Example 22. Let X be the cylinder over the surface $S=\{x^2y+x=z^2\}\subset\mathbb{A}^3=\mathrm{Spec}(\mathbb{C}[x,y,z])$, X is endowed with a hyperbolic $\mathbb{G}_m$-action given by a linear one on $\mathbb{A}^4=\mathrm{Spec}(\mathbb{C}[x,y,z,t])$, $\lambda(x,y,z,t)\to(\lambda^6x,\lambda^{-6}y,\lambda^3z,\lambda^2t)$.

Then X is equivariantly isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}^2,\mathcal{D})$ with:

$$\mathcal{D}=\left\{\frac{1}{2}\right\}D_1+\left\{\frac{1}{2}\right\}D_2-\left\{\frac{1}{3}\right\}D_3+\left[0,\frac{1}{\alpha_2\alpha_3}\right]E,$$

where E is the exceptional divisor of the blow-up $\pi:\tilde{\mathbb{A}}^2\to\mathbb{A}^2\simeq\mathrm{Spec}(\mathbb{C}[u,v])\simeq\mathrm{Spec}(\mathbb{C}[yt^3,yx])$, and where D_1, D_2 and D_3 are the strict transforms of the curves $L_1=\{v=0\}$, $L_2=\{1+v=0\}$ $L_3=\{u=0\}$ in $\mathbb{A}^2=\mathrm{Spec}(\mathbb{C}[u,v])$. This presentation is obtained using Remark 11. The divisor D_2 does not intersect the exceptional divisor E. Considering $\mathbb{S}(\tilde{\mathbb{A}}^2_{\mathrm{Supp}(D_2)},\mathcal{D}_{\mathrm{Supp}(D_2)})$ we obtain a $\mathbb{G}_m$-stable open neighborhood $\mathcal{V}=X_{1+xy}$ of the fixed point which is in addition equivariantly isomorphic to $\mathbb{A}^3_{1+Y(Z^{\alpha_2}+T^{\alpha_3})}$ in the same way as in Theorem 20.

4. Examples of non $\mathbb{G}_m$-rational varieties

Clearly the property to be G-uniformly rational is more restrictive than being only uniformly rational for a variety. It is therefore not surprising that there are smooth and rational varieties moreover equipped with a G-action which are not G-uniformly rational. In this section we will exhibit some varieties which are smooth and rational but not not $\mathbb{G}_m$-uniformly rational. However it is not known if these varieties are uniformly rational.

Let X be a smooth and rational variety endowed of an hyperbolic $\mathbb{G}_m$-action with a fixed point and with A-H quotient $Y(X)\simeq\tilde{\mathbb{A}}^2_{(u,v)}$ the blow-up of $\mathbb{A}^2=\mathrm{Spec}(\mathbb{C}[u,v])$ along the sub-scheme with ideal (u,v). Assuming moreover that in its presentation in terms of ps-divisor, it appears the strict transform of a smooth non rational affine curve through by the origin with a non integer coefficient. Then X is not $\mathbb{G}_m$-rational and so it is not $\mathbb{G}_m$-uniformly rational.

Proposition 23. *Let $C\subset\mathbb{A}^2$ be a smooth affine curve of positive genus through by the origin and let X be a $\mathbb{G}_m$-variety equivariantly isomorphic to $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)},\mathcal{D})$ with $\mathcal{D}=\left\{\frac{1}{p}\right\}D+[0,\frac{1}{p}]E$, where E is the exceptional divisor of the blow-up and D is the strict transform of C. Then X is a smooth rational variety but not a $\mathbb{G}_m$-uniformly rational variety.*

Proof. This is a direct consequence of the classification given in the corollary 12 of the hyperbolic $\mathbb{G}_m$-actions on $\mathbb{A}^3$. In this case the ps-divisors are rational. Moreover the variety $\mathbb{S}(\tilde{\mathbb{A}}^n_{(u,v)},\mathcal{D})$ are given in [21, proposition 3.1] and D is not rational. The varieties obtained by this construction are not $\mathbb{G}_m$-linearly rational and thus not $\mathbb{G}_m$-uniformly rational since the two properties are equivalent in the case of $\mathbb{G}_m$-varieties of complexity two. ☐

Example 24. Let $V(h)$ be a smooth affine curve of positive genus. Then the hypersurface $\{h(xy,zy)/y+t^p=0\}$, is stable in $\mathbb{A}^4=\mathrm{Spec}(\mathbb{C}[x,y,z,t])$, for the linear $\mathbb{G}_m$-action given by $\lambda\cdot(x,y,z,t)=(\lambda^px,\lambda^{-p}y,\lambda^pz,\lambda t)$. This variety is smooth and rational but not $\mathbb{G}_m$-uniformly rational.

4.1. **Numerical obstruction for rectifiability of curves.** The obstruction due to the genus of curve in the ps-divisor is not the only obstruction to being $\mathbb{G}_m$-rational. Indeed, there exist divisors $D=L_1+L_2$ where L_i is isomorphic to $\mathbb{A}^1$ for $i=1,2$ and such that D is not birationally equivalent to $D'=\{uv=0\}$. To see this, we will use a variant of the Kumar-Murthy dimension (see [20]). Recall that the pair (X,D) is said *smooth* if X is a smooth projective surface and D is a SNC divisor on X. For every divisor D on a smooth projective variety, we define the *Iitaka dimension,* $\kappa(X,D):=sup\,dim\phi_{|nD|}(X)$ in the case where $|nD|\neq\emptyset$ for some n, and $\kappa(X,D):=-\infty$ otherwise, where $\phi_{|nD|}:X\dashrightarrow\mathbb{P}^N$ is the rational map associated to the linear system $|nD|$ on X.

Lemma 25. *Let $D_0=\sum_{i=1}^k D_i$ be a reduced divisor on X_0 a complete surface, with D_i irreducible for each i. Let $\pi:X\to X_0$ be a resolution such that the strict transform D_X of D is SNC. Then the Iitaka dimension $\kappa(X,2K_X+D_X)$ does not depend on the choice of the resolution.*

Proof. By the Zariski strong factorization Theorem, it suffices to show that this dimension is invariant under blow-ups. Let (X,D_X) be a resolution of the pair (X_0,D_0) such that D_X is SNC. Let $\pi:\tilde{X}\to X$ be the blow-up of a point p in X. Since D_X is SNC, there are three possible cases, thus $p\notin D_X$ or p is only contained in an irreducible component of D_X or p is a point of intersection of two irreducible components D_X. It follows that $2K_{\tilde{X}}+D_{\tilde{X}}=\pi^*(2K_X+D_X)+(2-n)E$, $n=2,1,0$ respectively. Therefore,

$$\Gamma(X,\mathcal{O}(2K_{\tilde{X}}+D_{\tilde{X}}))=\Gamma(X,\mathcal{O}(\pi^*(2K_X+D_X)+(2-n)E))=\Gamma(X,\mathcal{O}(\pi^*(2K_X+D_X))),$$

and so by the projection formula ([14, II.5]), $\Gamma(X,\mathcal{O}(\pi^*(2K_X+D_X)))\simeq\Gamma(X,\mathcal{O}(2K_X+D_X))$. ☐

Definition 26. The *Kumar-Murthy dimension* $k_M(X_0, D_0)$ of (X_0, D_0) is the Iitaka dimension $\kappa(X, 2K_X + D_X)$ where $\pi : X \to X_0$ is any resolution such that the strict transform D_X of D_0 is SNC.

Definition 27. We say that a pair (Y, D) (as in definition 15) is *birationally rectifiable* if it is birationally equivalent to the union of $k \leq N = \dim(Y)$ general hyperplanes in $\mathbb{P}^N$. Note in particular that Y is rational and the irreducible components of D are either rational or uniruled.

By the previous proposition the Kumar-Murthy dimension is well defined independent on the choice of the resolution. Since, the Kumar-Murthy dimension of the pair $(\mathbb{P}^2, D)$, where D is a union of two lines, is equal to $-\infty$, we obtain:

Proposition 28. *If a reduced divisor $D = D_1 + D_2$ in $\mathbb{P}^2$ is birationally rectifiable then $k_M(\mathbb{P}^2, D) = -\infty$.*

Example 29. Let $C = \{u + (v + u^2)^2 = 0\}$ and $C' = \{\alpha v(v - \beta) + u = 0\}$ two curves in $\mathbb{A}^2 = \operatorname{Spec}(\mathbb{C}[u, v])$ where $(\alpha, \beta) \in \mathbb{C}^2$ are generic parameters chosen such that C and C' intersect normally. Let $D = \bar{C} + \bar{C}'$ be a divisor in $\mathbb{P}^2$ where $\bar{C}$ and $\bar{C}'$ are the closures of C and C' respectively. Then C and C' are isomorphic to $\mathbb{A}^1$ and $k_M(\mathbb{P}^2, D) \neq -\infty$.

Proof. The curve C' is clearly isomorphic to $\mathbb{A}^1$. In the case of C, consider the following two automorphisms: $\psi_1 : \begin{cases} u & \to u \\ v & \to v + u^2 \end{cases}$ and $\psi_2 : \begin{cases} u & \to u + v^2 \\ v & \to v \end{cases}$ then $\psi_2 \circ \psi_1 : \begin{cases} u & \to u + (u + v^2)^2 \\ v & \to v + u^2 \end{cases}$ gives that C is also isomorphic to $\mathbb{A}^1$. A minimal log-resolution of $\bar{C} \cup \bar{C}'$ is obtained by performing a sequence of seven blow-ups, five of them with centers lying over the singular point of $\bar{C}$ and the remaining two over the singular point of $\bar{C}'$.

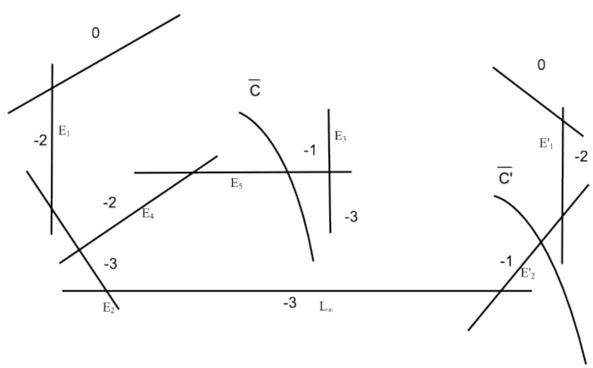

FIGURE 4.1. Resolution of $(\mathbb{P}^2, (\bar{C} + \bar{C}')$, the divisors E_i and E_i' are exceptional divisors obtained blowing-up $\bar{C} \cap L_\infty$ and $\bar{C}' \cap L_\infty$ numbered according to the order of their extraction.

Denote $\pi : S_7 \to \mathbb{P}^2$ the sequence of the seven blow-ups, and denote π_i for $i = 1 \dots 5$ the blow-ups for the resolution of $\bar{C}$ and π_i' for $i = 1 \dots 2$ the blow-ups for the resolution of $\bar{C}'$.

Thus, the canonical divisor of the surface S_7 is equal to $K_{S_7} = -3l + E_1 + 2E_2 + 3E_3 + 6E_4 + 10E_5 + E_1' + 2E_2'$, where l denotes the proper transform of general line in $\mathbb{P}^2$ while total transform of the divisor $\bar{C} + \bar{C}'$ is given by $\pi^*(\bar{C} + \bar{C}') = \bar{C} + 2E_1 + 4E_2 + 6E_3 + 11E_4 + 18E_5 + \bar{C}' + E_1' + 2E_2'$, where we have identified $\bar{C}$ and $\bar{C}'$ with their proper transforms in S_7.

Since $\bar{C}$ is of degree 4 and $\bar{C}'$ is of degree 2, the proper transform of $\bar{C} + \bar{C}'$ in S_7 is linearly equivalent to $6l$ and we obtain

$$2K_{S_7} + D = 2K_{S_7} + \Pi^*(\bar{C} + \bar{C}') - (2E_1 + 4E_2 + 6E_3 + 11E_4 + 18E_5 + E_1' + 2E_2') = E_4 + 2E_5 + E_1' + 2E_2',$$

which is an effective divisor. Thus $k_M(\mathbb{P}^2, D) \neq -\infty$, and by proposition 28, D is not birationally rectifiable.

□

4.2. **Application of the Kumar-Murthy dimension.** The previous sub-section allows to consider particular $\mathbb{G}_m$-threefolds and their presentation with the theory of ps-divisor. Let X be the subvariety of $\mathbb{A}^5$ given by:

$$\mathrm{Spec}(\mathbb{C}[w,x,y,z,t]/(w+y(x+yw^2)^2+t^{\alpha_3}, \alpha x(yx-\beta)+w+z^{\alpha_2})),$$

where $(\alpha,\beta)\in\mathbb{C}^2$ are the same parameters as in example 29. This variety is endowed with a hyperbolic $\mathbb{G}_m$-action induced by a linear one on $\mathbb{A}^5$, $\lambda\cdot(w,x,y,z,t)=(\lambda^{\alpha_2\alpha_3}w, \lambda^{\alpha_2\alpha_3}x, \lambda^{-\alpha_2\alpha_3}y, \lambda^{\alpha_3}z, \lambda^{\alpha_2}t)$. Moreover it is equivariantly isomorphic to

$$\mathrm{Spec}(\mathbb{C}[x,y,z,t]/(z^{\alpha_2}-\alpha x(xy-\beta)+y(x+y(z^{\alpha_2}-\alpha x(xy-\beta))^2)^2+t^{\alpha_3}).$$

Theorem 30. *Let X be the previous threefold, then X is a smooth rational variety but not a $\mathbb{G}_m$-uniformly rational variety.*

Proof. The A-H presentation of X is given by $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)},\mathcal{D})$ with

$$\mathcal{D}=\left\{\frac{a}{\alpha_2}\right\}D_1+\left\{\frac{b}{\alpha_3}\right\}D_2+\left[0,\frac{1}{\alpha_2\alpha_3}\right]E,$$

where E is the exceptional divisor of the blow-up $\pi:\tilde{\mathbb{A}}^2_{(u,v)}\to\mathbb{A}^2$, D_1 and D_2 are the strict transform of the curves C and C' as in example 29, in addition $(a,b)\in\mathbb{Z}^2$ chosen such that $a\alpha_3+b\alpha_2=1$. The presentation comes from the fact that X is endowed with an action of $\mu_{\alpha_2}\times\mu_{\alpha_3}$ factoring throught that of $\mathbb{G}_m$ and given by $(\epsilon,\xi)\cdot(x,y,z,t)\to(x,y,\epsilon z,\xi t)$.

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ X//\mu_{\alpha_2} & & X//\mu_{\alpha_3} \end{tikzcd}$$

By [21], $X//\mu_{\alpha_2}$ is equivariantly to $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)},\left\{\frac{1}{\alpha_3}\right\}D_2+\left[0,\frac{1}{\alpha_3}\right]E)$ and $X//\mu_{\alpha_3}$ is equivariantly to $\mathbb{S}(\tilde{\mathbb{A}}^2_{(u,v)},\left\{\frac{1}{\alpha_2}\right\}D_1+\left[0,\frac{1}{\alpha_2}\right]E)$. In fact, $X//\mu_{\alpha_2}$ and $X//\mu_{\alpha_3}$ are both isomorphic to $\mathbb{A}^3$ so that X is in fact a Koras-Russell threefolds (see [15, 17, 21]). By the computation in example 29, and proposition 28, we obtain the desired result.

□

5. Weak equivariant rationality

The property to be G-linearly uniformly rational is very restrictive. We will now introduce a weaker notion:

Definition 31. A G-variety X is called *weakly G- rational* at a point x if there exist an open G-stable neighborhood U_x of x, and $V\subset\mathbb{A}^n$ an open subvariety such that U_x is isomorphic to V. We said that X is *weakly G-uniformly rational* if it is weakly G- rational at every point.

In summary we have a sequence of implications between the different properties: G-linearly uniformly rational implies G-uniformly rational which implies G-weakly uniformly rational and this implies uniformly rational.

Theorem 32. *Let $S\subset\mathbb{A}^3=\mathrm{Spec}(\mathbb{C}[x,y,z])$ be the surface defined by the equation $z^2+y^2+x^3-1=0$, equipped with the restriction of the involution $(x,y,z)\to(x,y,-z)$ on $\mathbb{A}^3$.*

Then S is weakly μ_2-uniformly rational but not μ_2-uniformly rational.

Proof. The surface S is the cyclic cover of $\mathbb{A}^2$ of order 2 branched along the smooth affine elliptic curve $C=\{y^2+x^3-1=0\}\subset\mathbb{A}^2$. By construction, the inverse image of C in S is equal to the fixed points set of the involution. It follows that S is not μ_2 rational at the point $p=(1,0,0)$. Indeed, every μ_2-action on $\mathbb{A}^2$ being linearizable (see [18, Theorem 4.3]), its fixed points set is rational. Therefore there is no μ_2-stable open neighborhood of p which is equivariantly isomorphic to a stable open subset of $\mathbb{A}^2$ endowed with a μ_2-action. However, there is an open subest U of $\mathbb{A}^2$ which can be endowed with a μ_2-action such that U is equivariantly isomorphic to a μ_2-stable open neighborhood of p.

Let $u=z+y$ and $v=z-y$. Then S is defined by the equation $\{uv-x^3+1=0\}\in\mathbb{A}^3$. The open $V_1=S\setminus\{1+x+x^2=u=0\}$, isomorphic to $\mathbb{A}^2$ with coordinates u and $v/(1+x+x^2)=(x-1)/u=w$. The open subset $V=S\setminus\{1+x+x^2=0\}$ in V_1 is stable by μ_2 and contains the point p. Indeed, let $x=uw+1$ thus the coordinates ring of V is given by:

$$\mathbb{C}\left[u, w, \frac{1}{(uw+1)^2+uw+1+1}\right] = \mathbb{C}\left[u, w, \frac{1}{(uw)^2+3uw+3}\right].$$

The action of $\tau \in \mu_2$ the non-trivial element on V is given by:

$$\tau(u) = w((uw)^2+3uw+3); \quad \tau(w) = u((uw)^2+3uw+3)^{-1}.$$

The μ_2-stable open subset V of the surface S contains the fixed point p and it is isomorphic to an open subvariety of $\mathbb{A}^2$thus S is μ_2-weakly rational but not μ_2-rational. $\square$

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